Convective Cell Formation in a Z-Pinch

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Abstract

Closed field line confinement systems can develop convective cells when the MHD interchange stability criterion is violated. Using a previously derived set of reduced equations [V.P. Pastukhov and N.V. Chudin, Plasma Physics Reports **27**, 907 (2001)] it is shown that a true steady state solution can exist. For an assumed large-scale vortex pattern the plasma pressure profile that is implied by these convective flows as well as the non-local heat flux resulting from the convective flows is calculated .

Closed field line systems, such as a levitated dipole, provide a promising new direction for the magnetic confinement of plasmas for controlled fusion [1, 2]. A levitated dipole experiment known as LDX is presently under construction [3].

The plasma in a closed field line system can be stabilized in so-called "bad curvature" regions by plasma compressibility. In magnetohydrodynamic (MHD) theory, stability of interchange modes limits the pressure gradient to a value [4, 5]:

$$\frac{dln\ p}{dln\ U} < \gamma \tag{1}$$

with $U = \oint d\ell/B$ and γ is the ratio of specific heats, ($\gamma \equiv c_p/c_v = 5/3$ in three-dimensional systems at constant pressure and volume). When the pressure profile exceeds the flute mode stability limit the system may develop large-scale convective cells which can lead to nonlocal transport [6, 7, 8, 9, 10].

MHD stability can be simplified to produce a set of "reduced" MHD equations as has been developed by Kadomtsev and Pogutse [11] and by Strauss [12], which exclude fast magnetosonic waves from the MHD equations [13]. These equations, however, do not admit equilibrium steady state plasma flows. Pastukhov [14, 15] has recently developed a version of the reduced MHD set of equations which permits equilibrium flows and considers the convective cell drive for a pressure gradient that is close to and exceeds the marginal value given by Eq. (1). These equations were then applied to a hard core z pinch which can be considered as a model for a large aspect ratio levitated dipole configuration. Alfven waves, which have $m \neq 0$ with m the poloidal mode number, are excluded from consideration and he only considers $m = 0$ interchange modes. Pastukhov addresses the problem of pinch stability in the vicinity of the interchange stability boundary by solving an initial value problem and he finds that at sufficiently long times, large-scale convective cells will form and produce nonlocal energy transport. He finds solutions that exhibit large scale convective cells at long times although they are not in steady state. In this work it is assumed that the spatial dependence of the heating source is a function of the radial (flux) coordinate only.

In an actual experiment we would expect some variation of the heating in the symmetry coordinate [the axial (z) coordinate for a z pinch]. In this study we have utilized the Pastukhov equations and permitted a z variation of the heating source. We find that this set of equations has true steady state solutions.

We begin with the Pastukhov equations for a hard core z pinch. For simplicity we will assume a low beta plasma and use the vacuum field for the equilibrium field. The magnetic field dependence is $\vec{B} = \vec{e}_\theta B(r)$ and it is included in the MHD equations through the Jacobian variable, $J = \pi^{1/2}/U$, which for a z pinch becomes $J = \vec{B} \cdot \nabla\theta/2\sqrt{\pi} = B(r)/(2\sqrt{\pi}r)$. The plasma pressure ($p$) dependence is simplified through the use of the variable $S = p/J^\gamma$, which is related to entropy and is referred to as the entropy function.

Pastukhov developed the MHD equations, including weak dissipation terms, in the limit of a small parameter relating the transport and the MHD time scales, $\epsilon^3 \geq \chi/(ac_s)$ with χ the thermal diffusivity, c_s the sound speed and a the characteristic radial dimension of the system. He represented the entropy function as $S(r, z, t) = \overline{S}(t, r) + \tilde{S}(t, r, z)$ with

$\tilde{S}(t,r,z) \sim \epsilon^2 \overline{S}(t,r)$. The marginal stability condition can be shown to be $\overline{S}' = 0$ (the prime represents a radial derivative) and he assumed the system to be close to marginality, i.e. $\overline{S}'(t,r) \approx -\epsilon^2 \overline{S}(t,r)/a$. He then finds the adiabatic velocity field to be the $\mathbf{E}\times\mathbf{B}$ velocity, i.e.,

$$v_a = \frac{c}{J}[\nabla\theta \times \nabla\Phi] \sim (\rho_i/r)c_s \sim \epsilon\ c_s \tag{2}$$

with Φ the electrostatic potential, assumed to be $\Phi \sim T_e$ with T_e the electron thermal temperature and c_s the sound speed. The ion gyroradius is ρ_i and typically $\rho_i/r \sim 0.01$. We will further assume a radial mass density dependence $\rho \propto J(r)$. With this dependence the density will not be altered by convection and as a result the continuity equation decouples from the set of coupled equations that we will solve.

The advective terms in the Lagrangian derivatives $d/dt = (\partial/\partial t + v_a \cdot \nabla)$ give rise to non linear acceleration terms in the energy and momentum balance equations. If we keep the non-linear advective terms and appropriately order the energy and vorticity equations, the limit $\partial/\partial t \rightarrow = 0$, (i.e. , $d/dt \rightarrow (v_a \cdot \nabla)$) will produce zero-frequency modes that form convective cells. These equations represent perturbed equilibrium equations that include flows. Thus the Z-averaged equation for the entropy function becomes:

$$\frac{\partial \overline{S}}{\partial t} = 0 \rightarrow \frac{c}{rJ}\frac{d}{dr}\langle \tilde{S}\frac{\partial\Phi}{\partial z}\rangle - \frac{\gamma-1}{2J^\gamma}\frac{d}{dr}\left(r\rho\overline{\chi}\frac{d}{dr}\left(\frac{J^\gamma \overline{S}}{\rho}\right)\right) = \frac{\gamma-1}{J^\gamma}\overline{Q}_E \tag{3}$$

with

$$\langle f(r,z)\rangle \equiv \frac{1}{L}\int_0^L dz\ f(r,z) \tag{4}$$

where $\overline{Q}_E$ is the z-independent part of the heat source and the cyclic z-dimension is $0 < z < L$.

Equation (3) includes a thermal conduction term (proportional to $\overline{\chi}\nabla T$) which accounts for the heat loss that accompanies the setting up of the marginally stable profile and for the heat loss at the plasma edge where convection is expected to vanish (since the edge flow is parallel to the plasma boundary). The $\langle \tilde{S}\Phi_z\rangle$ term in Eq. (3) is responsible for the non linear convection-driven energy transport due to convective flows. Convective flows are seen to arise when $\overline{S}' < 0$ and the associated energy transport will prevent the pressure gradient from deviating significantly from the marginality condition, i.e. $\overline{S}' \sim O(\epsilon^2)$. Thus non local transport will determine the energy transport in the plasma core.

Keeping the fluctuating part of the source function, $\tilde{Q}_E$, the equation for the entropy function fluctuation, becomes:

$$\frac{\partial \tilde{S}}{\partial t} = 0 \rightarrow \frac{c}{rJ}\left[\Phi, \tilde{S}\right] - \frac{c}{rJ}\frac{d}{dr}\langle \tilde{S}\frac{\partial\Phi}{\partial z}\rangle + \frac{c}{rJ}\frac{\partial\Phi}{\partial z}\frac{d\overline{S}}{dr} = \frac{\gamma-1}{J^\gamma}\tilde{Q}_E(r,z) \tag{5}$$

with

$$[\Phi, f] \equiv \frac{\partial\Phi}{\partial z}\frac{\partial f}{\partial r} - \frac{\partial\Phi}{\partial r}\frac{\partial f}{\partial z}. \tag{6}$$

The square brackets derive from the advective acceleration terms. The $d\overline{S}/dr$ term in Eq. (5) results from the instability drive term for linear interchange instability. Ordinarily this term is large and when $d\overline{S}/dr < 0$ we obtain unstable growth described by $\partial\tilde{S}/\partial t \propto -d\overline{S}/dr$. Near to marginal stability, however, $d\overline{S}/dr \sim O(\epsilon^2)$ and the residual drive in Eq. 5 can be balanced by the non linear advection term in Eq. (5). When $d\overline{S}/dr \lesssim 0$ steady state requires that the instability drive be balanced by a beating of the flow with the pressure perturbation (in combination with an appropriate non axisymmetric heating profile). Thus the excitation of flows is seen to permit an equilibrium with an otherwise unstable pressure profile near the marginally linearly stable state. To solve Eq. (5) we can zero the perturbed heating function $\tilde{Q}_E(r,z)$ at an arbitrarily chosen location $z = z_0$, i.e.,

$$\tilde{Q}_E(r, z_0) = 0 \tag{7}$$

which will determine $\tilde{Q}_E(r,z)$ when $z \neq z_0$.

Following Ref. [14] we define a vorticity-like variable:

$$w = \nabla\theta \cdot (\nabla \times \frac{\rho v_a}{J}) = c\nabla \cdot \left(\frac{\rho \nabla \Phi}{r^2 J^2}\right) = c\nabla \cdot \left(\frac{\nabla \Phi}{\mu_0 \; c_A^2}\right) \tag{8}$$

with c_A the Alfven speed. Notice that this definition of vorticity is different from the more traditional definition because we have taken the curl of $\rho v_a/J$ rather than ρv_a. Taking $\nabla\theta \cdot Curl\{(MHD\ Momentum\ Equation)/J\}$ we obtain the resulting steady state vorticity equation:

$$\frac{\partial w}{\partial t} = 0 \rightarrow \frac{c}{r}\left[\Phi, \frac{w}{J}\right] + J^{\gamma-2}\frac{dJ}{rdr}\frac{\partial \tilde{S}}{\partial z} \approx 0. \tag{9}$$

In Eq. (9) we have left out terms on the right hand side that are small by ϵ^2 [14, 15]. Notice that the time rate of change of the vorticity is driven by the spatially fluctuating part of the entropy function.

Assuming a low beta plasma we can utilize the vacuum $J(r)$ value. If we guess at a simple form for $\Phi(r,z)$ we can then solve Eqs. (3), (5), (7), (8), and (9) for w, $\overline{Q}_E$, $\tilde{Q}_E$, $\overline{S}$, and $\tilde{S}$. As an example we choose Φ to include the two lowest harmonics in the cyclical coordinate, z:

$$\Phi(r,z) = \left(sin(\pi\frac{z}{L}) + sin(2\pi\frac{z}{L})\right) \; sin(\pi\frac{r - r_p}{r_w - r_p}) \tag{10}$$

with r_p, r_w the respective radial location of the pressure peak and the wall and L the cyclic z length. Figure 1 displays $\Phi(r,z)$. Equation (8) provides the vorticity w that is associated with this flow field and the vorticity equation, (Eq. 9), can be solved for the entropy fluctuation, $\tilde{S}$, which is shown in Fig. 2. The solution to these equations is analytic but tedious. They can however be easily solved using a computational tool such as Mathematica [16].

We can form the average $\langle\tilde{S}\partial\Phi/\partial z\rangle$, which is proportional to the average convective radial energy flow which appears on the left hand side of Eqs. (3) and (5). Again this result is analytic but too long to write here. In Eq. (3) the convective heat flow, $\langle\tilde{S}\partial\Phi/\partial z\rangle$, together with the thermal conductivity provide two energy flow channels that balance the heating source. Figure 3 displays the convective heat flux.

Next we utilize Eq. (5) to calculate the entropy ($\overline{S}(r)$) profile (to within an integration constant) and the fluctuating part of the heating function ($\tilde{Q}_E(r,z)$). We have the freedom to choose a z location at which $\tilde{Q}_E(r,z_0) = 0$. For each value of z_0 chosen, Eq. (5) will determine both the radial entropy profile, $\overline{S}(r)$, (which is independent of z) as well as the z-dependent function $\tilde{Q}_E(r,z)$. Figure 4 displays the average entropy function profile for $z_0 = L/4$. Recall that the ordering requires that $\overline{S} \approx constant$ to order ϵ^2. Figure 5 displays the fluctuating part of the heating source.

Since we are considering zero frequency modes, Fig. 3, which displays the convective cell generated heat flux, can be considered to be the excess heating, i.e., the heating profile that is in excess of the heat required to maintain a marginally stable profile. Combined with the z-dependent heating source, $\tilde{Q}_E$, these terms describe a two-dimensional heating profile that will drive the plasma convection profile that has been assumed in Eq. 10.

The solutions of the Pastukhov equations confirm the view that when a closed field line system that is stabilized by compressibility is driven past the stability limit for interchange modes, convective cells will form spontaneously as zero frequency modes. We have demonstrated the existence of true steady state solutions to the appropriate equations. Although we do not consider the stability of these solutions, we observe that experiments in closed field line systems with levitated internal coils have observed steady state convective cells [17].

In a dipole experiment we would expect the heating profile to have both a radial (i.e. flux) dependence and (weak) toroidal dependence. The ability of convective flows to remove excess heat, i.e., heating in excess of the amount necessary to maintain marginal stability, presents a major difference between closed field line confinement systems and systems that have rotational transform. In closed line systems the spontaneous development of convective cells can prevent systems that are driven past the marginally stable point from exhibiting violent, i.e., "disruptive" behavior.

We chose a particularly simple flow pattern and derived the implied hearing pattern. If a particular heating profile were imposed, as is the case in an actual experiment, the establishment of a time independent solution would require a complicated flow pattern and the system might not have a time independent solution.

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Figure Caption

1. Trial potential profile: $\Phi(r,z) = \left(sin(\pi\frac{z}{L}) + sin(2\pi\frac{z}{L})\right)\ sin(\pi\frac{r-r_p}{r_w-r_p})$
2. Entropy function fluctuation, $\tilde{S}(r,z)$.
3. Z-independent excess heating profile, $\overline{Q}_E(r)$
4. Z-independent entropy function profile, $\overline{S}(r)$
5. Fluctuation in the heat source, $\tilde{Q}_E(r,z)$ for $z_0/L = 0.25$

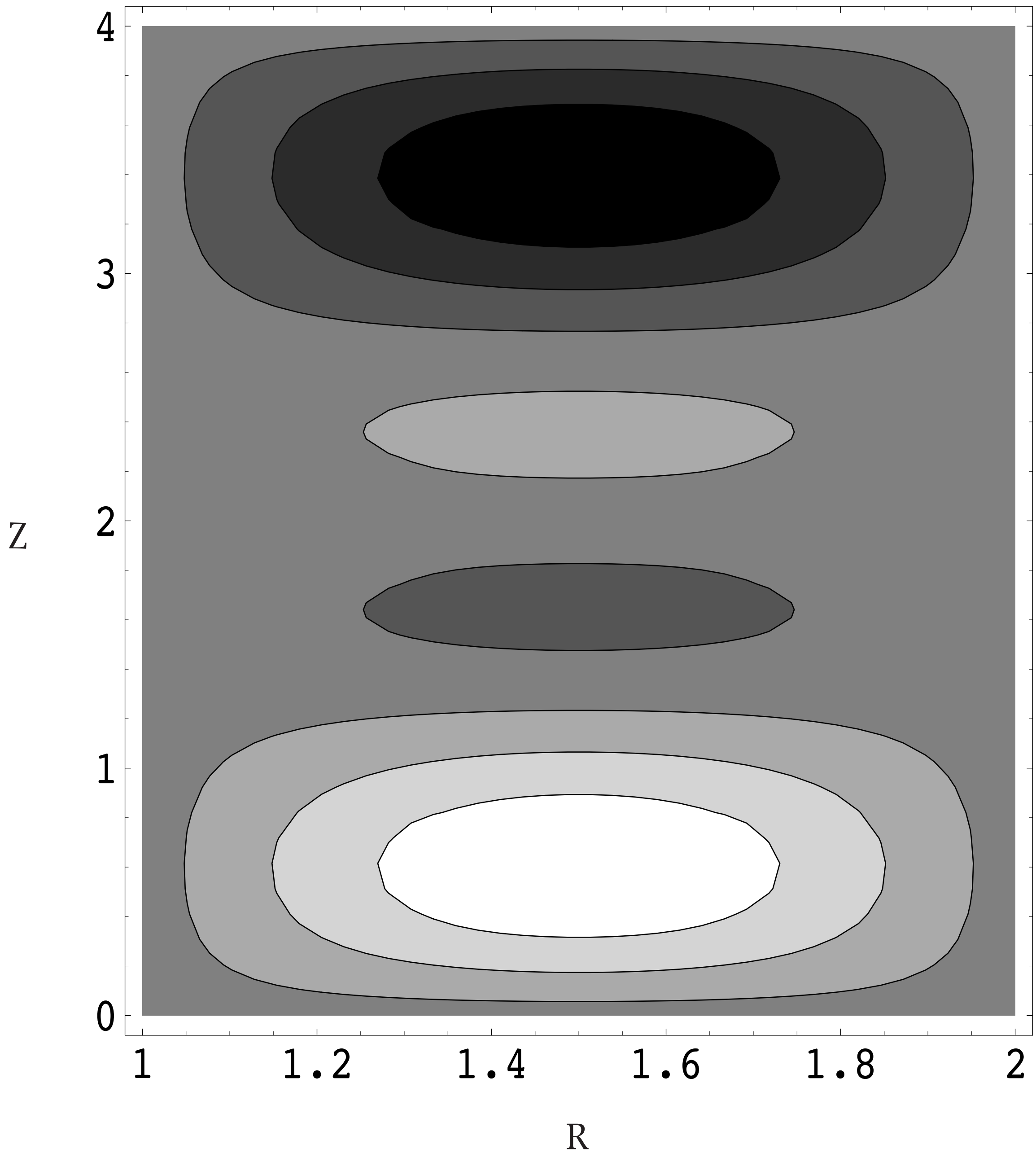

Fig 1. Potential Contours

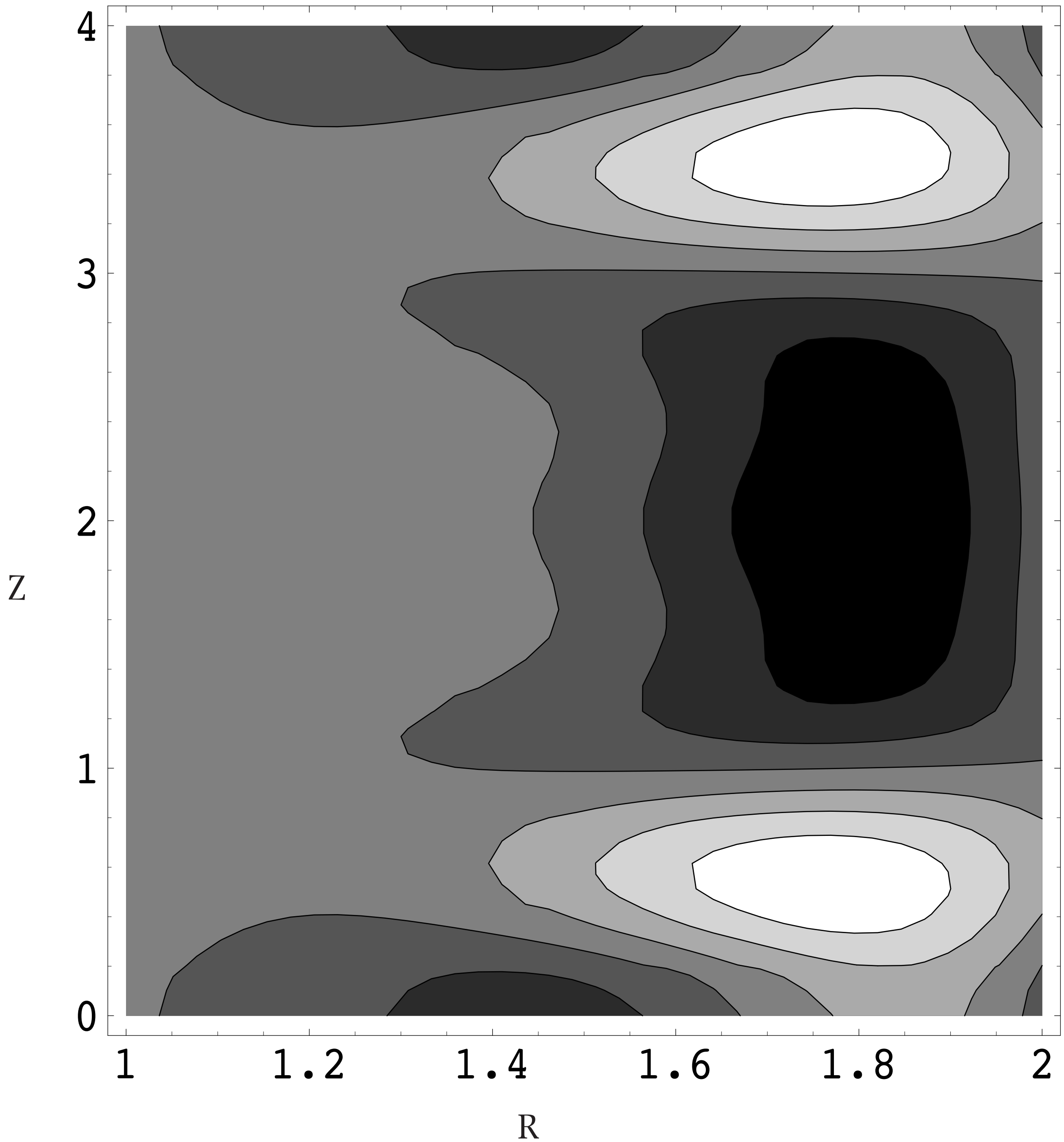

Fig 2. S fluctuation

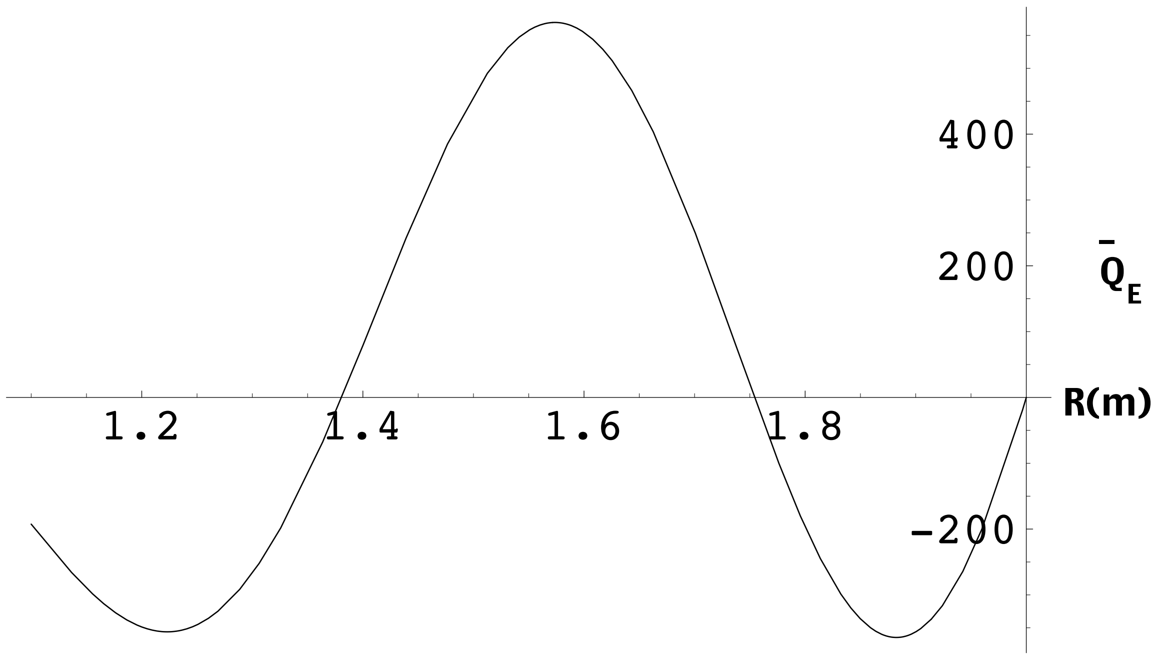

Fig. 3

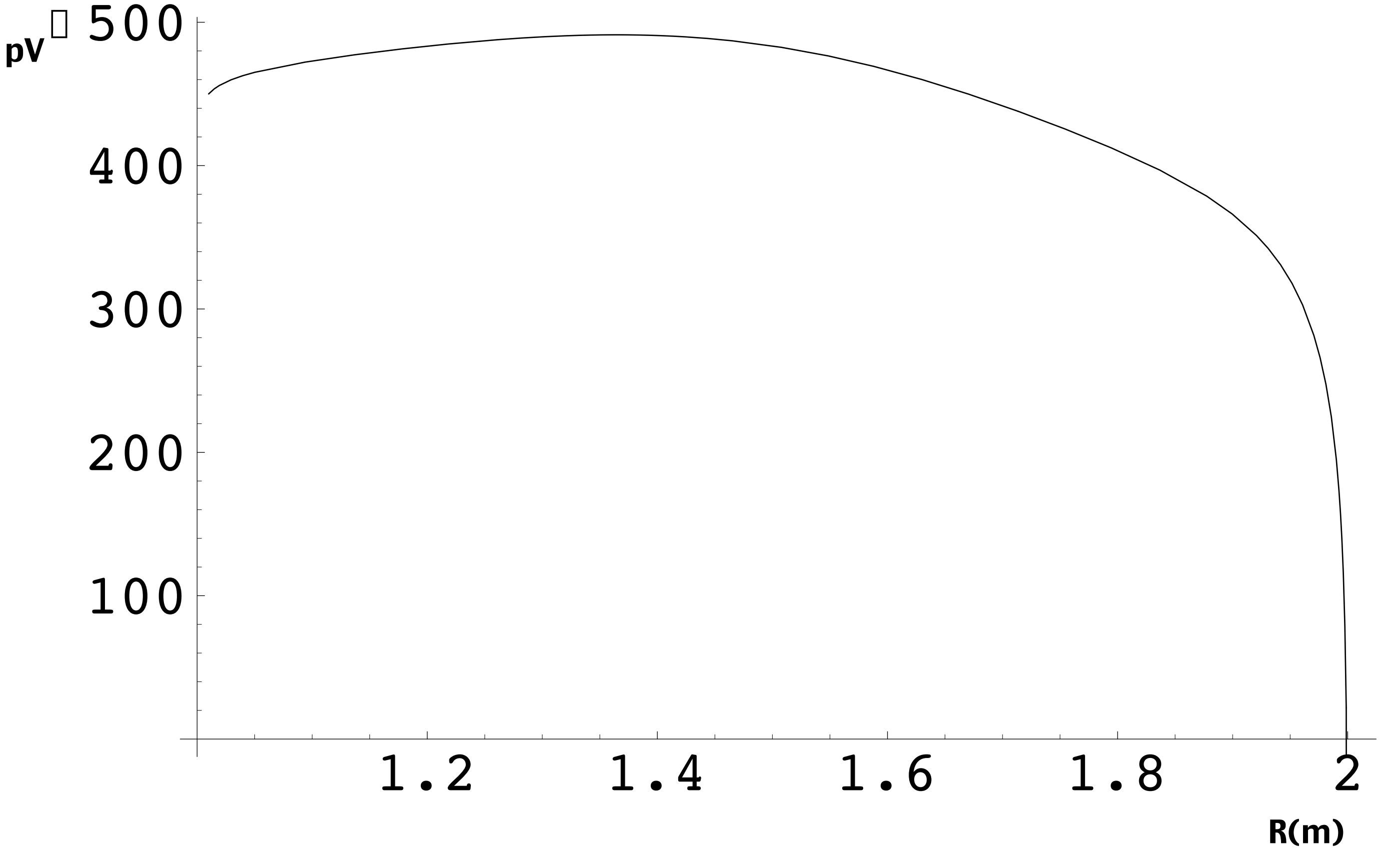

Fig. 4

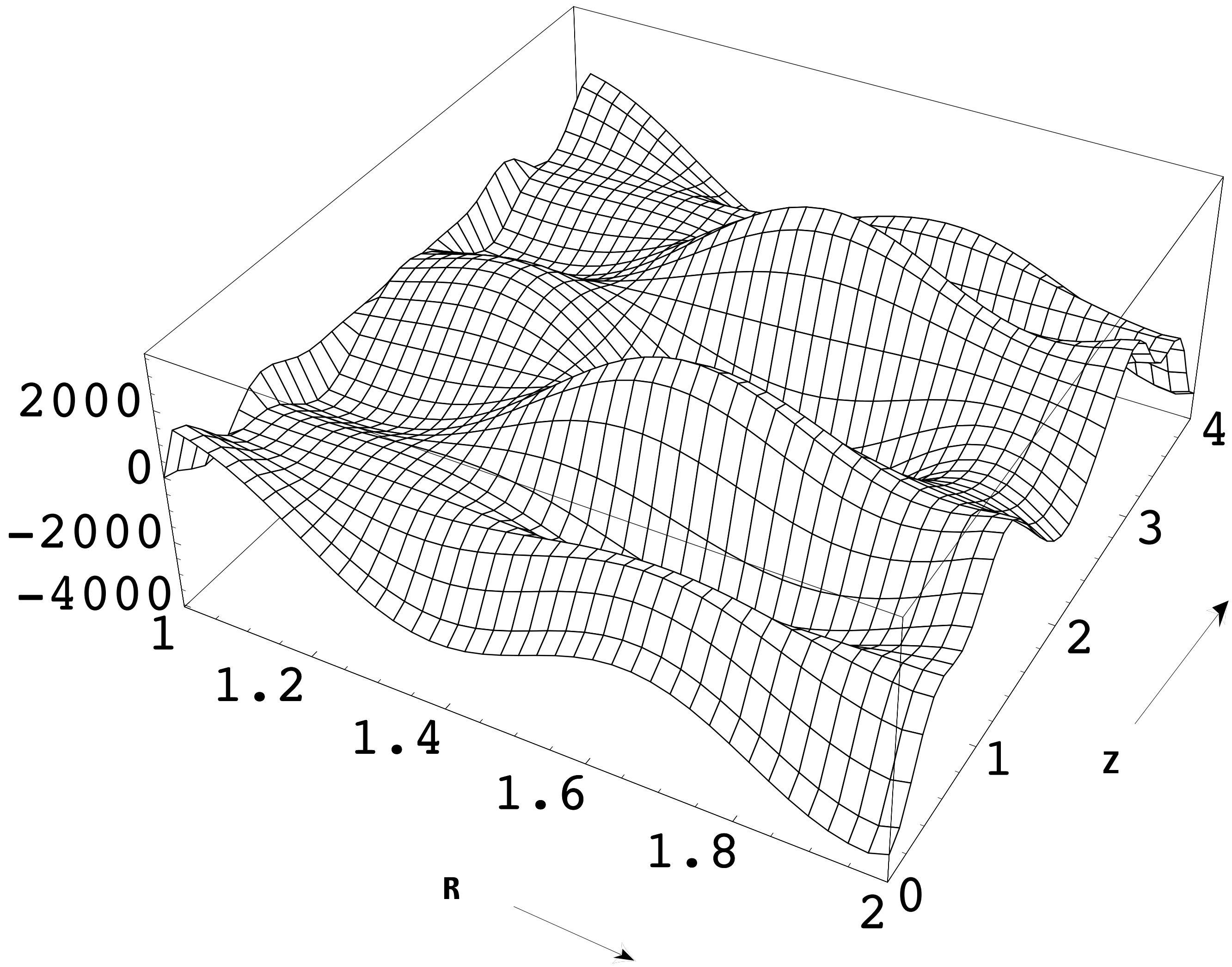

Fig. 5