Nonlinear Phenomena in RF Wave Propagation in Magnetized Plasma: a Review# *

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# This presentation is not a comprehensive review, most of the examples are from the author’s own work (or collaborators and students) and APOLOGIES to all the other experts, many of them here, who have made major contributions over the years

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Elements of Nonlinear Interaction of Waves in Plasma: Weak Turbulence Theory and Large Amplitude Coherent Waves

- Quasi-linear Theory: modification of the equilibrium distribution function with excited or externally injected waves (based on linear Landau Damping)
- Wave-wave scattering (interaction of 3 (or 4) waves - mode-mode coupling)
- Interaction of the beat of 2 waves with particles (nonlinear Landau Damping) and application to coherent wave interactions
- Applications to modern turbulence theories: nonlinear mode coupling and zonal flow generation
- Application to RF wave heating: Parametric Decay Instabilities (PDI) and soliton formation near resonances: pump wave depletion
- Application to laser fusion: Parametric Decay Instabilities (PDI): generation of energetic particles and pump wave depletion
- Strong turbulence not discussed here: particle trapping in large amplitude waves; particle orbit modification (diffusion) etc
Radio Frequency Waves of Interest for Heating and Current Drive Cover a Wide Range of Frequencies

Radio Frequency Spectrum

- **ALFVÉN**
- **ICW, FW, IBW**
- **LH Wave (TRIVELPIECE-GOULD)**
- **O, X, EBW**

RF sources ➔ 2.5 MW tetrode 0.7 MW klystron 1 MW gyrotron

Frequency (GHz)
Quasilinear Theory: Landau Damping of waves deform the equilibrium distribution function

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{q_j}{m_j} \left( \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial f}{\partial \mathbf{v}} + \frac{q_j}{m_j} \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0,
\]

\[
\nabla \cdot \mathbf{E} = 4\pi \sum_i q_j \int d^3 \mathbf{v} f_i(\mathbf{v}),
\]

\[
f(\mathbf{x}, t) = g(\mathbf{x}, t) + \sum_{k \neq 0} f_k \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]
\]

\[
\frac{\partial g}{\partial t} = -\frac{q_j}{m_j} \sum_{k'} E_{-k'} \frac{\partial f_{k'}}{\partial \mathbf{v}},
\]

\[
\frac{\partial f_k}{\partial t} + \mathbf{v} \frac{\partial f_k}{\partial \mathbf{x}} + \frac{q_j}{m_j} E_k \frac{\partial g}{\partial \mathbf{v}} = -\frac{q_j}{m_j} \sum_{k \neq 0} E_{k-k'} \frac{\partial f_{k'}}{\partial \mathbf{v}},
\]

\[
\imath k E_k = 4\pi \sum_i q_j \int d\mathbf{v} f_k.
\]
The quasilinear diffusion equation is given by

\[ \frac{\partial g}{\partial t} = \frac{\partial}{\partial \nu} \left( D \frac{\partial g}{\partial \nu} \right), \]

where

\[ D = \left( \frac{q}{m} \right)^2 \sum \frac{|E_{k'}|^2}{(i k' \nu - i \omega)} \]

\[ \lim_{\delta \to 0} + \frac{1}{\omega_k - k \nu + i \delta} = \frac{P}{\omega_k - k \nu} - i \pi \delta(\omega_k - k \nu), \]  

Landau pole

\[ D_R = 8\pi \left( \frac{q}{m} \right)^2 \int dk \epsilon_k \pi \delta(\omega - k \nu), \]  

Resonant particles

\[ D_{NR} = 8\pi \left( \frac{q}{m} \right)^2 \int dk \frac{\epsilon_k \gamma_k}{(\omega - k \nu)^2}, \]  

Nonresonant particles (sloshing energy)

where \( \epsilon_k = |E_k|^2 / 8\pi \) is the wave energy

Finally, the WKB equation

\[ \partial \epsilon_k / \partial t = 2\gamma_k \epsilon_k. \]
Lower Hybrid Current Drive based on Quasi-linear theory: asymmetric LH wave packet creates one sided plateau on the electron distribution whose velocity moment corresponds to net current *(Fisch and Karney, ‘78)*

**Maximum** \(n_{\|}\) penetrates until it is Landau damped on electrons at the **quasi-linear plateau** break point, typically \(v_{\min} = c/n_{\|}\) \(max = 2.4v_{Te}\) where \(v_{Te} = \sqrt{2T_e/m_e}\), and populates the plateau,

\[
n_{\|_{\text{max}}} = 7/\sqrt{T_e}(\text{keV})
\]

**Minimum accessible** value of \(n_{\|}\) determines **maximum** value of the quasi-linear plateau, \(v_{\max} = c/n_{\|_{\text{min}}}\) which determines maximum current drive efficiency, \(\eta_{\text{CD}}\)

Thus, **window of penetration** is limited to

\[
\frac{\omega_{pe}}{\omega_{ce}} + \left[1 + \frac{\omega_{pe}^2}{\omega_{ce}^2}\left(1 - \frac{\omega_{ce}\omega_{ci}}{\omega^2}\right)\right]^{1/2} \leq n_{\|} \leq 7/\sqrt{T_e(\text{keV})}
\]

Or for \(\omega_{pe}^2/\omega_{ce}^2 \approx 1\) about 12 keV !

**Higher magnetic field** is beneficial!
Stix solved the Fokker Planck equation with the Kennel-Engelmann quasi-linear operator in the steady state for ICRH.

\[ \frac{\partial f(v)}{\partial t} \approx \frac{\pi Z^2 e^2}{8m^2|k||} |E_x + iE_y|^2 \sum_n \frac{1}{v} \frac{\partial}{\partial v} v^2 |J_{n-1} \left( \frac{k v}{\omega_{ci}} \right) |^2 \]

\[ \delta[v - \left( \omega - n\omega_{ci} \right) \frac{k}{v}] \frac{1}{v} \frac{\partial f}{\partial v} \]

\[ + C(f(v)) \]

**POWER ABSORBED**

\[ P = \int d^3v \frac{mv^2}{2} \frac{\partial f(v)}{\partial t} \]

**SINGLE PASS ATTENUATION**

\[ 2\eta = \frac{P}{S} \]

where \( S \) is the Poynting flux.
Minority ion cyclotron heating in a two-ion species plasma was described quantitatively in the 1970s by Stix’s theory.

Fokker-Planck Energy Distribution Calculated for RF Excitation at the Minority Ion Cyclotron Frequency

\[ \xi \propto \left( \frac{m<P>T_e^{1/2}}{n_e n Z^2} \right) \]

Hosea et al., Nuclear Fusion, 1975

PLT Hydrogen Ion Energy Distribution Compared with Theory

\[ P_{RF} = 350 \text{ kW}, \text{ D-H Plasma, } n_e = 2 \times 10^{13} \text{ cm}^{-3}, \]
\[ I_P = 300 \text{ kA} \]

Hosea et al., Varenna Workshop, 1979
Minority heating verified by mass sensitive charge exchange neutral diagnostic in the TM-1-Vch Tokamak by V. L. Vdovin and coworkers at the Kurchatov Institute (1976, Grenoble)


Fig. 13. Signal from charge exchange neutral detector at $U = 1$ keV.

Fig. 14. Build of a high energy "tail" in the proton (1%) energy distribution in deuterium (99%) for two levels of HF excitation at $\omega = \omega_B^H$.
In memory of Victor Vdovin, renowned Russian plasma physicist and expert in RF physics

20th Topical Conference on RF Power in Plasmas
Sorrento, Italy, 2013

(a) Resonant mode-mode coupling

(b) Nonlinear Landau interaction

(i) Nonlinear wave–particle (Compton) scattering

(ii) Scattering from shielded (dressed) particle
Third order perturbation theory is needed to calculate both resonant and non-resonant mode-mode coupling

First-order solution:

\[ \epsilon(\omega, k) \phi_{k}^{(1)}(t) = 0. \]

Second-order solution:

\[ \epsilon(\omega, k) \phi_{k}^{(2)}(t) = - \sum_{j} \sum_{k'} \frac{4 \pi q^3 n_0}{k^2 m^2} \int d^3 v \int_{0}^{\infty} d\tau \int_{0}^{\infty} d\tau' G_k(\tau) \left[ (k - k') \frac{\partial}{\partial v(\tau)} G_{k'}(\tau') k' \frac{\partial g}{\partial v(\tau')} + k' \frac{\partial}{\partial v(\tau)} G_{k-k'}(\tau')(k-k') \frac{\partial g}{\partial v(\tau')}ight] \phi_{k'} \phi_{k-k'} \]

Third-order solution:

\[ \epsilon(\omega, k) \phi_{k}^{(3)}(t) = - \sum_{j} \sum_{k'} \frac{4 \pi q^3 n_0^2}{k^2 m^2} \int d^3 v \left( \int_{0}^{\infty} d\tau \int_{0}^{\infty} d\tau' G_k(\tau)(k-k') \frac{\partial}{\partial v(\tau)} G_{k'}(\tau') k' \frac{\partial g}{\partial v(\tau')} \left[ \phi_{k-k'}^{(2)} \phi_{k'} + \phi_{k-k'} \phi_{k-k'}^{(2)} \right] \right) \\
+ \sum_{k''} \frac{i q}{m} \int_{0}^{\infty} d\tau \int_{0}^{\infty} d\tau' \int_{0}^{\infty} d\tau'' G_k(\tau)(k-k') \frac{\partial}{\partial v(\tau)} G_{k'}(\tau')(k-k'') \\
\times \frac{\partial g}{\partial v(\tau')} G_{k-k''}(\tau'') k'' \frac{\partial g}{\partial v(\tau'')} \phi_{k-k'} \phi_{k-k'} \phi_{k''} \phi_{k''} \right) .
\]

where \( G_k(\tau) = \exp[i k (x' - x) + \omega \tau] \)

\[ x' = x - v \tau, \quad \tau = t - t' \]
The Mode Coupling Equation
(both resonant and no-resonant terms)

Here $\epsilon(\omega, k)$ is the linearized quasilinear dispersion relation, namely,

$$\epsilon(\omega, k) = 1 + \sum_i \frac{\omega_i^2}{k^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dv \frac{\partial g/\partial v}{(\omega - kv)}$$

Using the WKB expansion,

$$\gamma \phi \approx \partial \phi/\partial t \text{ (Aamodt, 1965)}$$

we obtain the following mode coupling equation:

$$\frac{\partial \epsilon(\omega, k)}{\partial \omega} \frac{\partial \phi_k(t)}{\partial t} = \gamma_k \frac{\partial \epsilon(\omega, k)}{\partial \omega} \phi_k + \sum_{k'} iQ_{k-k', k'} \phi_{k-k'} \phi_{k'}$$

$$+ \text{Im}[P_{b-b'} \phi_b \phi_{b'}^* |^2 + R_{b-b-b'} \phi_{b-b'} \phi_{b-b'}^* |^2 + S_{k-k-k'} \phi_k \phi_{k-k'} |^2],$$

where the term proportional to $Q_{k-k', k'}$ represents resonant mode-mode coupling, and the terms proportional to $P, R, S$ represent nonlinear Landau damping.
Resonant Mode-Mode Coupling (3 wave interaction)  
Fixed Phase Waves  
(Kadomtsev, Sagdeev, Aamodt, etc)

\[ iS_{k}(dA_{k}/dt) = V_{k,k',k''}A_{k'}A_{k''}, \]

\[ iS_{k'}(dA_{k'}/dt) = V_{k,k',k''}^{*}A_{k''}A_{k'}, \]

\[ iS_{k''}(dA_{k''}/dt) = V_{k,k',k''}^{*}A_{k'}A_{k''}, \]

where we have defined

\[ A_{k}(t) = \phi_{k}(t)[(k^{2}/8\pi)|\partial \epsilon / \partial \omega |]^{1/2} \]

\[ \Delta \omega = -\omega_{k} + \omega_{k'} + \omega_{k''} = 0, \]

\[ \Delta k = -k + k' + k'' = 0, \]

\[ V_{k,k',k''} \]

\[ = \frac{1}{H} \sum_{J} \frac{4\pi e_{J}^{3}}{m_{J}^{2}} \int_{-\infty}^{\infty} d^{3}v \frac{1}{\omega_{1} - k_{1}v + i\delta} \]

\[ \times \left[ k_{2} \frac{\partial}{\partial v} \frac{k_{3} \partial f_{ij}}{\omega_{3} - k_{3}v + i\delta} \right] \delta_{k_{1} k_{2}} \]

\[ \delta_{k_{1} k_{2}} \]

where

\[ H = 16\pi \left| \frac{k^{2}}{8\pi} \frac{\partial \epsilon_{k}}{\partial \omega_{k}} \frac{k'^{2}}{8\pi} \frac{\partial \epsilon_{k'}}{\partial \omega_{k'}} \frac{k''^{2}}{8\pi} \frac{\partial \epsilon_{k''}}{\partial \omega_{k''}} \right|^{1/2}. \]

Symmetry relations:

\[ V_{k,k',k''} = V_{k,k'',k'} = V_{-k,-k',-k''}, \]

\[ V_{k,k',k''} = V_{-k',-k,k''} = V_{-k'',k',-k}. \]
Stability of Resonant 3 Wave System

\[ \frac{d^2 A_r}{dt^2} = - |V_{k, k', k''}|^2 \left( s_k s_{k'} |A_{k''}|^2 + s_k s_{k''} |A_{k'}|^2 \right) \Delta \]

\[ \frac{d^2 A_{k'}}{dt^2} = - |V_{k, k', k''}|^2 \left( s_k s_{k'} |A_{k''}|^2 - s_k s_{k''} |A_{k'}|^2 \right) \Delta \]

\[ \frac{d^2 A_{k''}}{dt^2} = - |V_{k, k', k''}|^2 \left( s_k s_{k''} |A_{k'}|^2 - s_k s_{k'} |A_{k''}|^2 \right) A_{k''}. \]

- If at least one (but not all) of the waves is negative energy \((s < 0)\) explosive instability results (ie, all wave amplitudes grow without bound)
- If all waves have same energy, stability results
- If one wave is considerably larger amplitude than the others (Pump Wave) a decay instability results until the pump wave energy is depleted – a complete analog of the Parametric Decay Instability (but now with finite pump wave-vector); then initial growth rate is
  \[ \gamma \propto \frac{|V_{k, k', k''}| |A_0|}{|A_{k'}|^2} \]
- Characteristic pump depletion time is
  \[ t_0 \approx \frac{1}{2V_{k, k', k''} |A_0|} \ln \frac{A_{0}(0)}{A_{1}(0)} \]
Random Phase Waves - 3 Wave Interaction in Turbulence Theories

\[
\langle \phi_k \phi_k^* \rangle = |\phi_k|^2 \delta(k - k'), \quad \langle \phi_k \rangle = 0, \quad \langle E^2 \rangle = \int |A_k|^2 dk = \int N_k dk.
\]

\[
\frac{d}{dt} N_k = 4\pi_k \sum_{k'} |V_{k,k',k''}|^2 \delta_{k,k'+k''} \delta(\omega_{k} - \omega_{k'} - \omega_{k''})
\]

\[
\times [N_{k'} N_{k''} - N_{k'} N_{k''} S_{k} S_{k'} - N_k N_{k'} S_{k'} S_{k''}] + 2\gamma_k N_k
\]

where \(2\gamma_k N_k\) is due to quasilinear growth. Equation (2.32) may be repeated for \(N_{k'}\) and \(N_{k''}\) by rotating indices. We note that the terms \(N_{k'} N_{k''}\) and \(N_k N_{k'}\) on the right-hand side are due to spontaneous emission, and for an unstable system may be neglected.

Used in turbulence theories, including nonlinear Gyrokinetic theory; we shall not discuss this equation any further in this talk.

Instead, we concentrate on fixed phase coherent waves that can be launched and analysed in the laboratory.
Resonant Mode-Mode Coupling of Bernstein Waves in the presence of one large amplitude Bernstein Wave (Pump wave) in the case of co-linear propagation (after Porkolab and Chang, 1970)

\[
\frac{dE_1(x)}{dx} + \left[ \alpha_1 + \frac{\gamma}{(\partial \omega/\partial k)_1} \right] E_1(x) = \frac{\Gamma_1 E_n(x) E_2(x)}{(\partial D/\partial \omega)_1 (\partial \omega/\partial k)_1},
\]

\[
\frac{dE_2(x)}{dx} + S \left[ \alpha_2 + \frac{\gamma}{(\partial \omega/\partial k)_2} \right] E_2(x) = S \frac{\Gamma_2^* E_1(x) E_0(x)}{(\partial D/\partial \omega)_2 (\partial \omega/\partial k)_2},
\]

\[
\Gamma_{k, k', k''} = \sum_{p,n} \frac{\left( \omega_{p,n}^2 \right)}{\omega_c^2} \left( \frac{e}{m} \right)
\cdot \int_0^\infty dv_\perp \left( \frac{\partial f_0}{\partial v_\perp^2} \right) J_p(a v_\perp) J_{p-n}(b v_\perp) J_n(c v_\perp) \Theta,
\]

\[
\Theta = \left[ \frac{1}{(\Omega - p)^2 - 1} \right] \left[ \frac{n/k'''}{\Omega'' - n} + \frac{(p - n)/k'}{\Omega - (p - n)} \right]
\]

\[
\Omega = \omega / \omega_c
\]

\[
\frac{dE_0(x)}{dx} + \alpha_0 E_0(x) = 0,
\]

\[
D(k, \omega) = 1 + \frac{1}{\kappa^2 \lambda^2 D} \sum_{n=1}^\infty \frac{2n^2 \exp(-b I_n(b))}{n^2 - \Omega^2}
\]

Porkolab_RF_2015-Lake Arrowhead-Ca
Nonlinear decay instability of electron Bernstein waves: growth rate in quantitative agreement with theory (after Chang and Porkolab, 1970)

FIG. 5. (a) Typical decay spectrum; $f_1 = 335$ MHz, $f_2 = 420$ MHz, $f_0 = 758$ MHz. The pump signal ($f_0$) is greatly attenuated by a filter. $\omega_c$ (cyclotron frequency). (b) Dispersion relation and decay modes. Each set of three identical symbols represents a pair of decay modes like that indicated by the arrows. There are seven pairs of decay modes shown in the figure. (After Chang and Porkolab, 1970.)
Nonlinear Landau Damping: Beat of two waves resonant with particles via Landau interaction

Wave-wave particle resonance selection rules: the beat wave resonates with particles:

\[
\frac{dN_k}{dt} = 2\gamma_k N_k + \sum_{k''} S_{kk''} L_{k,k''} N_{k''} N_{k''},
\]

\[
\frac{dN_{k''}}{dt} = 2\gamma_{k''} N_{k''} - \sum_{k'''} S_{k''k'''} L_{k,k''} N_k N_{k'''}
\]

\[
L_{k,k''} = \sum_j \int d^3\nu 2\pi \delta(\omega' - k'\nu) \frac{\omega'^2}{k'^2} \left| \frac{\partial\epsilon(\omega', k')}{\partial\omega'} \right| k' \frac{\partial g}{\partial\nu} \tilde{L}_{k,k''},
\]

where

\[
\tilde{L}_{k,k''} = \left| \frac{2V_{k,k'',k'}}{\epsilon(\omega', k')} - \frac{2e_k k'^2 (kk'')}{m_j (\omega - kv)(\omega'' - k''v)H} \right|
\]

The Diffusion coefficient is given by:

\[
D_{NLLD} = \frac{4\pi}{M^2} \int d^3k \int d^3k'' \frac{N_k N_{k''} 4\pi \omega^4 k(\omega' - kv')}{|\partial\epsilon/\partial\omega| |\partial\epsilon/\partial\omega''| k^2 k''^2 M n_0} \left| \frac{kk''}{(\omega - kv)^2} - \frac{H^2 V_{k,k'',k'''} n_n}{k''^2 e M} \right|^2
\]
Nonlinear Cyclotron Damping for Bernstein Waves with small but finite $k_{\parallel}$ (after Porkolab and Chang, Phys. Fluids, 1972)

Symmetry rules were proven for arbitrary values of the Larmor radius. Similar results were obtained for the resonant mode-mode coupling coefficient. The integrals were performed numerically and compared favorably with experiment!
Solution of the Nonlinear Landau Coupled Equations for Two Narrow Wave Packets if $\gamma_k = \gamma_{k^*}$

\[ N_k(t) = \frac{\exp\{2\gamma_k t - Z [1 - \exp(2\gamma_k t)]\}}{1 + [S_{k^*} N_k(0)/(\tilde{N}_{k^*}(0) + S_{k^*} N_k(0))]}(\exp[-L (1 - \exp(2\gamma_k)) + 1]) \]

where

\[ Z = \frac{(L_{k, k^*}/2\gamma_k)(N_{k^*}(0) + S_{k^*} N_k(0))}{2\gamma_k} \]

It is easy to see that for one negative energy wave (i.e., $S_{k^*} = -1$) and one positive energy wave (i.e., $S_k = 1$) and $L_{k, k^*} > 0$ explosive instability results at time

\[ t^*_e = \frac{1}{2\gamma_k} \ln \left[ 1 + \frac{2\gamma_k \ln(N_{k^*}(0)/N_k(0))}{L_{k, k^*}(N_{k^*}(0) - N_k(0))} \right] \]

Note that in the limit $\gamma_k = \gamma_{k^*} = 0$, 

\[ t^*_e = \frac{\ln(N''(0)/N(0))}{L_{k, k^*}(N''(0) - N(0))} \]

Decay instability if one wave is of large amplitude - pump wave - leads to Parametric Decay Instability (PDI) – verified with EBW

Porkolab_RF_2015-Lake Arrowhead-Ca
Nonlinear Landau (Cyclotron) Damping of Bernstein Waves observed in laboratory experiments

Large amplitude electron Bernstein wave decays into another Bernstein wave and the beat wave resonates with particles at $n\omega_c$

Similar interaction observed later in IBW experiments in tokamaks
Amplification of a Bernstein test wave by inverse nonlinear cyclotron damping in the presence of a large amplitude Bernstein Wave allowed measurement of the nonlinear coupling coefficient.

Quantitative measurement of wave amplitudes and growth rates in excellent agreement with predictions of theory – verified nonlinear coupling coefficient to arbitrary $kr_c$ (Chang and Porkolab, Phys Fluids, 1970, 1972.)
Plasma Wave Echoes verified the long term memory of ballistic particles due to wave interaction – not the same as Landau damping-(Gould and O’Neil, Malmberg and Wharton, Porkolab and Sinnis, etc, 1967-69)

Second order echo position is predicted to be at $l^* = l\omega_2/(\omega_2 - \omega_1)$ corresponding to the solid lines

Due to ballistic terms in Landau poles and not the same as NLLD
Cyclotron echoes show memory of wave particle interaction with electron Bernstein waves at cyclotron harmonics

Cyclotron Echo position is

\[ z^* = d(\omega_2 - n\Omega) / (\omega_2 - \omega_1 - p\Omega), \]

**FIG. 22.** (a) Echo power as a function of cyclotron frequency \( f_c \). (b) Interferometer output of echo power as a function of receiver position. For both (a) and (b), \((f_1, f_2, f_3) = (391, 700, 309)\) MHz; transmitters at \( z_{1,2} = 0, 30 \) cm. (After Porkolab and Sinnis, 1968.)

**FIG. 23.** \( z^* \) vs \( d \). The numbers in braces are \((f_1, f_2, f_3, f_4)\). Solid lines show best fit to experimental data, giving \( z^*/d = s \). Theoretical lines, according to Eq. (4), show agreement to within 1% or better by assuming that \((n, p) = (2, 1), (3, 1), (3, 2)\) for the lines with \( s = 2.30, 2.00, 1.54 \), respectively. (After Porkolab and Sinnis, 1968.)
Damping of ion wave echoes by ion-ion Coulomb collisions in agreement with theoretical predictions

\[ \frac{n_e}{n_0} = \frac{l}{\lambda_1} \exp\left[-\alpha \left( \frac{\lambda_1 \lambda_3}{\lambda_2 \lambda_{mfP}} \right)^{1/5} \left( \frac{l}{\lambda_1} \right)^{3/5} \right] \]

where \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) are the wavelengths of the primary waves (1, 2) and the echo (3), \( \lambda_{mfP} \) is the mean free path for collisions, and \( l \) is the transmitter separation.

(Wong and Baker, 1969)
Theory of parametric decay instabilities in the presence of a large amplitude coherent pump wave

- Theory applicable to magnetically confined fusion plasmas, or laser plasma fusion
- Theory usually applicable to coherent waves (but ignores particle trapping in large amplitude finite wavelength pump wave - long pump wavelength)
- Decay waves may be two normal modes of the system, or one normal mode and a “quasi-mode” due to Landau (cyclotron) damping at the beat frequency (analogous to nonlinear Landau damping discussed before - but theory much simpler - no need for 3rd order perturbation theory since \( k_0 \approx 0 \))
- Of interest is substantial growth in space or time to cause pump depletion, or significant particle acceleration (hot electrons in laser fusion)
- Mode coupling caused by the “quiver velocity” of electrons oscillating in the large amplitude pump wave field but stationary ions - but may be extended to the ion cyclotron regime and driven by the relative motion of different ions
- For large \( E_{\parallel} \) may cause significant density depletion (ponderomotive force)
- May result in “soliton” formation - a concentrated hump of electric field of finite spatial extent “digging” a density cavity to form a long lived structure – especially if the frequency is near plasma resonances (\( \omega_{\text{UH}}, \omega_{\text{LH}}, \omega_{\text{ii}} \))
Physical picture of parametric decay instabilities (PDI) in the presence of $k=0$ rf pump electric field

**Electron “quiver” motion:** \( \Delta x \approx (eE_0/\omega_0^2) \cos \omega_0 t \)

Electrons beat with background ion fluctuations

\[
\delta n_e \approx \frac{\partial n_i}{\partial x} \Delta x = ikn_i \frac{eE_0}{m \omega_0^2} \cos \omega_0 t
\]

Consider now the harmonic oscillator equation for electron plasma waves \( \omega_{ek} = \omega_{pe} + 3k^2v_{te}^2 \) with additional \( \delta n_e \)

\[
\frac{\partial^2 n_e}{\partial t^2} + \nu_e \frac{\partial n_e}{\partial t} + \omega_{ek}^2 n_e = \frac{\partial^2 (\delta n_e)}{\partial t^2} = \frac{ikn_i e E_0}{m} \cos \omega_0 t.
\]

Now consider the inverse, beating of the pump field with the electron plasma wave, producing a ponderomotive force \((\mathbf{v} \cdot \nabla \mathbf{v})\) at the ion frequency, where we used Poisson’s equation;

But ion acoustic waves are driven by electron pressure:

\[
\omega_s^2 n_i = \frac{k_i^2 T n_i}{m} = \frac{k^2 (p + \delta p)}{m},
\]

So the ion acoustic temporal response is:

\[
\frac{\partial^2 n_i}{\partial t^2} + \nu_s \frac{\partial n_i}{\partial t} + \omega_s^2 n_i = -\frac{iken_i e E_0}{m_i} \cos \omega_0 t,
\]
By Fourier transforming, we find that instability results above the threshold field

$$\frac{\gamma_e \gamma_s}{\omega_{ek} \omega_s} < \frac{E_0^2}{64 \pi n_0 T_e}$$

Inhomogeneous plasma threshold

$$K(X) = \Delta K = k_0 - k_1 - k_2 \quad I = I_0 \exp\left(\frac{2 \pi \gamma_s^2}{|K'|} v_{1x} v_{2x}\right)$$

where the normal mode frequencies are $$\omega_1 \approx \omega_s$$ and $$\omega_2 = \omega_{ek}$$ and the linear damping rates are $$\nu_s / 2 = \gamma_s$$ and $$\gamma_c \approx \nu_e / 2$$ and the upper sideband was ignored (off resonance).

In addition, it was shown by Nishikawa that a purely growing mode (also called oscillating two stream instability) is also possible if we keep both upper and lower sideband and let the low frequency mode has a zero frequency:

$$\omega_{ek} \approx (\omega_0 \pm \omega_1), \text{ for } \Re \omega_1 = 0 \text{ we obtain the following threshold:}$$

$$\frac{\gamma_e}{\omega_0} \langle \frac{E_0^2}{32 \pi n_0 T_e} \rangle$$

**Soliton Formation**

If we consider strong electric fields, the ponderomotive force may deplete the equilibrium density, forming an ambipolar potential, and the ions respond:

$$\nabla_x p_e + \nabla_x \left( \frac{1}{8\pi} |E|^2 \right) = ne \nabla_x \phi, \quad \nabla_x p_i = -ne \nabla_x \phi, \quad n(x) \approx n_0 \exp\left[ -\frac{|E| - \langle |E|^2 \rangle}{16\pi n (T_e + T_i)} \right]$$
Consider now the high frequency response:

\[ \frac{\partial^2 E}{\partial t^2} + \omega_e^2 E + v_e \frac{\partial E}{\partial t} = 0, \]

recognizing that \( \omega_e^2 = \omega_{pe}^2 (1 + 3k^2 \lambda_D^2) \) and that \( \omega_{pe} \propto n(x) \), we may perform a WKB-type expansion of \( E \), that is, assume \( E = E_H(t, x) \exp(-i\omega_0 t) \) (where \( E_H \) is a slowly varying function of \( x \) and \( t \)).

It follows that

\[ i \frac{\partial E_H}{\partial t} + \frac{3v_e^2}{2\omega_0} \frac{\partial^2 E_H}{\partial x^2} + \frac{\omega_{pe}^2}{2\omega_0} \frac{|E_H|^2}{16\pi n T_e} E_H = 0, \quad (4.31) \]

where \( v_{te}^2 = T_e/m_e \), and where the last term of Eq. (4.31) contains the cubic nonlinearity. This is the so-called nonlinear Schrödinger equation. It is easy to show that in the linearized limit Eq. (4.31) predicts the purely growing mode.

M. Porkolab and M. Goldman derived the NL Schroedinger Equation for Upper hybrid Solitons (Phys Fluids 19, 872 (1976)) that was verified experimentally by T. Cho and S. Tanaka (Phys. Rev. Lett. 45, 1403 (1980))
Laboratory experiments demonstrate evidence of ponderomotive force in an unmagnetized plasma

(Wong and Stenzel, 1975)

FIG. 45. Space-time representation of ion bursts (shaded) driven by the ponderomotive force. Density cavities are created as a result of ion expulsion. The polarization of ion trajectories is represented by the cone with a half-width of 40° with respect to the z axis. (After Wong and Stenzel, 1975.)
Thresholds due to plasma inhomogeneities or finite spatial pump wave extent usually dominate.

The effects of density gradients can be obtained following the procedure outlined by Rosenbluth 1972. Assuming a WKB-type phase variation and defining

\[ K(X) = \Delta K = k_0 - k_1 - k_2 \]

(where we assumed a one-dimensional propagation so that all \( k \)'s depend on \( x \) only), the equations describing the spatial variation of the two coupled modes can be written in the following form (Rosenbluth, 1972).

\[
\begin{align*}
\frac{dE_1}{dx} + \frac{\gamma_1}{v_{1x}} E_1 &= \frac{\gamma E_2^*}{v_{1x}} \exp \left( i \int_0^x K(x) dx \right), \\
\frac{dE_2}{dx} + \frac{\gamma_2}{v_{2x}} E_2 &= \frac{\gamma E_1^*}{v_{2x}} \exp \left( -i \int_0^x K(x) dx \right),
\end{align*}
\]

(4.20a)

(4.20b)

where \( E_1, E_2 \) are the electric field amplitudes of the decay waves, and the other quantities have been defined earlier. Assuming a linear variation of the mismatch with distance, namely \( K(x) \sim K'x \), Eqs. (4.20a) and (4.20b) can be combined in the following form

\[
\frac{d^2 \psi}{dx^2} - \left[ \frac{\xi^2}{4} + \frac{1}{2} \frac{d \xi}{dx} + \frac{\gamma_0^2}{v_1 v_2} \right], \psi = 0,
\]

(4.20c)

where \( \xi = \gamma_2/v_{2x} - \gamma_1/v_{1x} - iK(x) \), and \( \psi \propto E_1 \). Integrating

\[ I = I_0 \exp \left( 2\pi \gamma_0^2 / |K'| v_{1x} v_{2x} \right), \]

(4.21)

where linear damping has been ignored. Thus the effective threshold may be defined as the pump power for which the initial background noise is amplified by a factor of \( 2\pi \), namely,

\[ 1 < \frac{\gamma_0^2}{|K'| v_{1x} v_{2x}}. \]

(4.22)

For example, for decay into weakly damped ion acoustic waves an electron plasma waves one finds for the threshold

\[ \frac{v_0^2}{v_t^2} > \frac{16}{(k^2 \lambda_0^2 \rho / H)^4/3}. \]

(4.23)

However, for backward scattering \((v_1 \cdot v_2 < 0)\) absolute instability may occur if

\[ \gamma_0^2 > \frac{v_1 v_2}{4} \left[ \left( \frac{\gamma_1}{v_1} - \frac{\gamma_2}{v_2} \right)^2 + \frac{\pi^2}{L^2} \right], \]

(4.19)

where \( L \) is the length of the system, \( v_1, v_2 \) are the group velocities of the two decay waves, \( \gamma_1, \gamma_2 \) are their damping rates, and \( \gamma \) is the homogeneous, uniform plasma growth rate (see, for example, Pesme et al., 1973).
Parametric instabilities in Laser Fusion (examples of PDI in an un-magnetized plasma)

noteworthy processes include Raman scattering (EM → EM + E.P.), decay into two electron plasma waves (EM → E.P. + E.P.), Compton (induced) scattering (EM → EM + particles), and filamentational and modulational instabilities (where EM designates the electromagnetic wave, and E.P. designates longitudinal electron plasma waves). Thresholds and growth rates for these processes have been given in the literature.
Parametric Instabilities in a Magnetized Plasma: “weak” pump wave (analogous to previous theory)

(M. Porkolab, Phys Fluids, 1974)

\[ E_0(x, t) = (E_{0x} \hat{x} + E_{0z} \hat{z}) \cos \omega_0 t, \]

\[ \frac{\partial f}{\partial t} + \hat{x} \cdot \nabla_x f + \frac{q}{m} \left( \frac{v \times B}{c} + \hat{E}_o \cos \omega_0 t \right) \cdot \nabla_v f = 0 \]

Electrons are displaced relative to ions according to the relations.

\[ \Delta x = \frac{e}{m} \frac{E_{0x} \cos \omega_0 t}{\omega_0^2 - \Omega^2}, \]

\[ \Delta y = \frac{\Omega}{\omega_0} \frac{e}{m} \frac{E_{0x} \sin \omega_0 t}{\omega_0^2 - \Omega^2}, \]

\[ \Delta z = \frac{e}{m} \frac{E_{0z} \cos \omega_0 t}{\omega_0^2}, \]

\[ \varepsilon(\omega) + \frac{\mu^2}{4} \chi_i(\omega)[\chi_e(\omega)] \left[ \frac{1}{\varepsilon(\omega - \omega_0)} + \frac{1}{\varepsilon(\omega + \omega_0)} \right] = 0, \]

where we assumed \( \varepsilon(\omega - \omega_0) \ll 1, \mu^2 \ll 1. \) Here

\[ \varepsilon(\omega \pm j\omega_0) = 1 + \chi_i(\omega \pm j\omega_0) + \chi_e(\omega \pm j\omega_0) \]

is the linear dielectric function, \( j = 0, \pm 1, \) and \( \chi_i(\chi_e) \) is the linear ion (electron) susceptibility. These susceptibilities can be written quite generally in terms of the complete hot plasma dielectric tensor in a magnetic field, including collisions. The coupling coefficient \( \mu \) is given by the expression

\[ \mu = \frac{e}{m} \left[ \left( \frac{E_{0x} k_x}{\omega_0^2} + \frac{E_{0x} k_x + E_{0y} k_y}{\omega_0^2 - \Omega^2} \right)^2 + \frac{(E_{0y} k_y - E_{0y} k_x)^2 \Omega^2}{(\omega_0^2 - \Omega^2)^2 \omega_0^2} \right] \]

Coupling due to parallel drift, polarization drift and ExB drift.
PDI theory in magnetized plasma - Cont.

Resonant Decay Instability

\[
(\gamma + \gamma_1)(\gamma + \gamma_2) = \frac{\mu^2}{4} \frac{\chi_4(\omega_1)\chi_a(\omega_2)}{\theta \omega} \left| \frac{\partial \varepsilon_1}{\theta \omega_1} \right| \left| \frac{\partial \varepsilon_2}{\theta \omega_2} \right|,
\]

where we neglected the upper sideband. Here \(\gamma_1, \gamma_2\) are the linear damping rates, and

\[
\text{Re} \varepsilon_j(\omega_j, k_j) = 0
\]
defines the normal modes \(\omega_j(k_j), j = 1, 2\). We note that these normal modes satisfy the selection rules [Eq. (4.3)]. The threshold is obtained from Eq. (4.14) by setting \(\gamma = 0\).

The purely growing mode is obtained by assuming \(\omega_1 = 0, \varepsilon_R(\omega_2, k) \approx 0\), in which case the growth rate near threshold is (Porkolab, 1974)

\[
\gamma = -\frac{\mu^2}{4k^2\chi_4^2(\omega_2)\chi_a(\omega_2)(1 + T_i/T_e)}
\]

so that \(\gamma \propto E_0^2\) (since \(\mu \propto E_0\)). For large electric fields (i.e., for \(\gamma \gg k_n v_t, \gamma \gg k_c\)), we obtain

\[
\gamma = \frac{\mu^2/3}{\omega_0^2} \left( \frac{\partial \varepsilon_R(\omega_2, k)/\theta \omega_0}{\theta \omega_0} \right)^{1/3} (1 + T_i/T_e)^{1/3}
\]

so that \(\gamma \propto E_0^{2/3}\), where \(\omega_2 \approx \omega_0\).

Non-resonant PDI into quasi-modes

Decay into quasimodes is obtained by assuming a resonant mode at the lower sideband [so that \(\varepsilon_R(\omega - \omega_0) = \varepsilon_R(\omega_0) = 0\)] and a nonresonant mode at the low-frequency response so that \(\varepsilon_R(\omega) \neq 0\). The growth rate is (Porkolab, 1977)

\[
\gamma = \frac{\mu^2}{4} \left[ (|x_4(\omega)|^2 + x_{1R}(\omega)x_{eR}(\omega) + (|x_4(\omega)|^2 + x_{eR}(\omega)x_4^*(\omega)) x_{4R}(\omega)) \right] \frac{\partial \varepsilon/\partial \omega_2}{\varepsilon(\omega)^2}
\]

The above Eq. predicts particularly strong instabilities when the wave-particle resonance conditions

\[
\frac{(\omega_0 - \omega_2)}{k_n} \approx v_{te},
\]

\[
\frac{(\omega_0 - \omega_2)}{k_n} \approx n\Omega/k_n \approx v_{ti},
\]

are satisfied. These are similar conditions to nonlinear Landau damping discussed earlier but here the derivation is greatly simplified since we took \(k_0 = 0\), ie, spatially uniform pump wave.
There are also processes in which two electromagnetic waves couple. For example, the decay of a whistler wave into another whistler wave and an ion acoustic wave is an example of so-called Brillouin scattering, in which an incident electromagnetic wave decays into another electromagnetic wave and an ion acoustic wave (Forslund et al., 1972; Porkolab et al., 1972). In this case the coupling mechanism is the $j \times B$ force, i.e., particles oscillating in the electric field of one of the waves produce a current $j$, which then couples with the magnetic field component $B$ of the second electromagnetic wave. The $j \times B$ force then gives a contribution to electron pressure which may drive electrostatic ion waves unstable. This process is of the backscattered type, namely,

$$k_0 = -k_1, k_2 = 2k_0,$$

where $k_2$ is the wave vector of the ion wave.
<table>
<thead>
<tr>
<th>Pump wave</th>
<th>Decay wave</th>
<th>Threshold</th>
</tr>
</thead>
</table>
| Extraordinary mode (if ignore tunneling) \((E_1 B_0)\); \(\omega_0 = \omega_{\text{UH}}\) \((\omega_{\text{pe}} = \Omega_0)\) | Lower-hybrid | \[
\frac{\omega_{\text{pe}}}{(2 \Omega_0 \omega_1)^{1/2}} \left(\frac{\gamma_1 \gamma_2}{\omega_1 \omega_2}\right)^{1/2} \leq \frac{E_0}{8 (\pi n_0 T_e)^{1/2}}
\] |
| Extraordinary mode (if ignore tunneling) \((E_1 B_0)\); \(\omega_0 = \omega_{\text{UH}}\) \((\omega_{\text{pe}} = \Omega_0)\) | Upper-hybrid | \[
\frac{\omega_{\text{pe}}}{\omega_1} \left(\frac{\gamma_1 \gamma_2}{\omega_1 \omega_2}\right)^{1/2} \leq \frac{E_0}{8 (\pi n_0 T_e)^{1/2}}
\] |
| Ordinary mode \((\omega_0 = \omega_{\text{pe}})\) | Ion acoustic | Electron plasma |
| Electron plasma wave (Trivelpiece–Gould) | Ion acoustic | Electron plasma |
| Whistler wave \((\omega_0 < \Omega_0 < \omega_{\text{pe}})\) | Ion acoustic | Electron plasma |
| Lower-hybrid wave (resonance cone, whistler wave) \(k_{\parallel} / k \approx 3 (m_e / m_i)^{1/2}\) | Lower-hybrid | \[
\frac{\Omega_0 \omega_0}{2 \omega_{\text{pe}}^2} \left[2 \frac{\gamma_1 \gamma_2}{\omega_2} \left(1 + \frac{\omega_{\text{pe}}^2}{\Omega_0^2}\right)\right]^{1/2} \leq \frac{E_0}{8 (\pi n_0 T_e)^{1/2}}
\] |
| Magnetoacoustic wave \((\omega_0 \approx \Omega_i)\) | Ion cyclotron | Drift wave |

\[
2 (\omega_0 \Delta \Omega_i)^{1/2} \leq \frac{k c E_0}{B}
\]
Possible PDI processes in the presence of intense ECH from pulsed FELs in tokamak plasmas

(Porkolab and Cohen, Nuclear Fusion, 1988)

**FIG. 1.** (a) Schematic of parametric processes for an ordinary wave, \( \omega_0 \approx \Omega_e(0) \), incident from the outside of the torus. PDI denotes parametric decay instability. (b) Schematic of parametric processes for an extraordinary wave, \( \omega_0 \approx 2\Omega_e(0) \), incident from the outside of the torus. \( eB \) denotes the electron Bernstein wave and \( eC \) denotes the electron cyclotron wave.

### TABLE II. PARAMETRIC INSTABILITIES FOR ORDINARY MODE HEATING, \( \omega_0 \approx \Omega_e(0) \)

<table>
<thead>
<tr>
<th>Instability</th>
<th>( \omega - \hat{\kappa} ) matching satisfied</th>
<th>Importance, comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflective</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Raman scattering by upper hybrid waves</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>2. Raman scattering by lower hybrid waves</td>
<td>Yes</td>
<td>No, convective stabilization</td>
</tr>
<tr>
<td>3. Brillouin scattering by ion cyclotron wave or</td>
<td>Yes</td>
<td>Unstable, may be important</td>
</tr>
<tr>
<td>4. Brillouin scattering by ion quasi-mode</td>
<td>Yes</td>
<td>Unstable, may be important</td>
</tr>
<tr>
<td>5. Brillouin scattering by ion Bernstein waves</td>
<td>Yes</td>
<td>No, convective stabilization</td>
</tr>
<tr>
<td>Forward scattering</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Ponderomotive force at plasma surface</td>
<td>(Not applicable)</td>
<td>No, ( \Delta t ) pulse too short</td>
</tr>
<tr>
<td>7. Thermal and ponderomotive filamentation</td>
<td>Yes</td>
<td>No, convective growth length much larger than the system</td>
</tr>
<tr>
<td>Absorptive</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. Parametric decay to upper hybrid wave and ion quasi-mode</td>
<td>Yes</td>
<td>No, convective stabilization</td>
</tr>
<tr>
<td>9. Oscillating two-stream four-wave interaction</td>
<td>Probably not</td>
<td>No, requires ( \omega_0 &lt; \omega_{uh} ) and has very weak growth rates</td>
</tr>
<tr>
<td>10. Parametric decay to two magnetized plasma waves</td>
<td>Yes</td>
<td>Probably important if ( 1/4 &lt; \Omega_{p}^2 / \Omega_{e}^2 &lt; 1 )</td>
</tr>
</tbody>
</table>
Observation of PDI in laboratory experiments in the ECH regime (Greko and Porkolab, PRL 1973)

**FIG. 1.** Dispersion relation for Bernstein waves for $\omega_p = \omega_c$. $R$ is the electron Larmor radius; the dotted line shows effects of finite $k_\parallel$. Insets (a) to (c) show typical decay spectra. The dots are associated with decay into lower hybrid waves, and the squares with ion-acoustic waves. $\omega_p/\omega_c = 2.12$ for spectrum (a), 2.08 for (b), and 1.50 for (c).

**FIG. 3.** (a) Electron energy distribution $|F_0(\omega)|$ for different powers ($P$) 5 $\mu$sec after the start of the heating pulse. $\omega_p/\omega_{ce} = 1$, $\omega_p/\omega_{ce} \approx 1.5$. 

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PDI observed during X-Mode ECH launch (inside launch) on the Versator II tokamak near the upper hybrid frequency with weak single pass absorption (S. McDermott, M. Porkolab, et al, PF, 1982)

\[ f_0 = 35.08 \text{ GHz} \]

Bernstein wave sideband

\[ f_1 = 34.68 \text{ GHz} \]

Hot plasma lower hybrid wave sideband

\[ \omega_1^2 = \omega_{\text{uh}}^2 - \omega_{pe}^2 b_1 \]

\[ \omega_2^2 = \omega_{\text{lh}}^2 [1 + (3b_2 \omega_{ce} \omega_{ci} T_i)/(\omega_2^2 T_e)] \]

Convective threshold power in good agreement with theory

\[ \frac{(E_x)_{th}^2}{64 \pi n_e T_e} = \frac{1}{4} \frac{\omega_{pe}^2}{\omega_{ce}^2} \frac{2 \pi}{k_1 \Delta x} \]
More complete theory of PDI, valid for larger pump wave amplitudes, including $\mu \approx 1$ (M. Porkolab, Phys. Fluids 1974, 1977)

\[ \frac{\partial f}{\partial t} + \nabla \cdot \nabla f + \frac{a}{m} \left( \frac{\nabla \times B}{c} + E + \frac{E_0}{c} \cos \omega_0 t \right) \cdot \nabla f = 0 \]

\[ F(\chi, \phi, t) = \frac{a}{m} \int_0^t dt' \nabla \cdot \phi \cdot \phi \cdot \nabla f \exp \left[ i \mu \sum_j \sin (\omega_j t - \beta_j) \right] \]

\[
1 + \frac{1}{\chi_i} = J_0^2(\mu) \frac{\chi_e}{1 + \chi_e} + J_1^2(\mu) \left( \frac{\chi_e^+}{1 + \chi_e^+} + \frac{\chi_e^-}{1 + \chi_e^-} \right) \\
+ \frac{J_0^2(\mu) J_1^2(\mu)}{1 + (1/\chi_i)} \left[ \frac{\chi_e/(1 + \chi_e)}{1 + \chi_e} - \frac{\chi_e^+/1 + \chi_e^+}{} \right]^2 \\
+ \frac{J_0^2(\mu) J_2^2(\mu)}{1 + (1/\chi_i)} \left[ \frac{\chi_e^-/(1 + \chi_e^-)}{1 + \chi_e^-} - \frac{\chi_e^-/(1 + \chi_e^-)}{1 + \chi_e^-} \right]^2 \\
+ \frac{J_0^2(\mu) J_1^2(\mu)}{1 + (1/\chi_i)} \left[ \frac{\chi_e^-/(1 + \chi_e^-)}{1 + \chi_e^-} - \frac{\chi_e^-/(1 + \chi_e^-)}{1 + \chi_e^-} \right]^2
\]

\[
\mu = \frac{e}{m} \left[ \left( \frac{E_{0z} k_{||}}{\omega_0^2} + \frac{E_{0z} \cdot k_{\perp}}{\omega_0^2 - \Omega_e^2} \right)^2 + \frac{[(E_0 \times k_{\perp}) \cdot \hat{z}]^2 \Omega_e^2}{(\omega_0^2 - \Omega_e^2)^2 \omega_0^2} \right]^{1/2}
\]

Used this equation to analyze numerically many different decay modes in magnetized plasma; then if applicable, compare with small $\mu$ expansion of coupled mode equations and evaluate convective and inhomogeneous plasma thresholds
PDI of Lower Hybrid Waves - early experimental observation in tokamaks on ATC in 1977 (M. Porkolab et al, PRL 1977)

FIG. 1. (a)-(d) Parametric-decay spectrum due to a split waveguide coupler; D₂ gas; $\bar{n} = 1.8 \times 10^{13}$ cm⁻³. (a) $P_{in} = 65$ kW, (b) $P_{in} = 41$ kW, (c) $P_{in} = 19$ kW, and (d) $P_{in} = 1.5$ kW; $f_{ci} = 13$ MHz. (e),(f) Parametric-decay spectrum due to a single waveguide; H₂ gas; $\bar{n} = 1.2 \times 10^{13}$ cm⁻³, $P_{in} = 35$ kW. (e) Low-frequency spectrum, and (f) pump and sideband; $f_{ci} \approx 25$ MHz.

FIG. 2. (a) Sideband amplitude vs $P_{in}$ for a split-wave-guide coupler; D₂ gas. Curve A, $\bar{n} = 1.8 \times 10^{13}$ cm⁻³; curve B, $\bar{n} = 1.4 \times 10^{13}$ cm⁻³; and curve C, $\bar{n} = 1.1 \times 10^{13}$ cm⁻³. For $\bar{n} < 9 \times 10^{12}$ cm⁻³ and $P \leq 100$ kW, no decay is observed. (b) $\Delta T_{\perp}$ and decay-wave amplitude vs $P_{in}$; H₂ gas; $\bar{n} = 1.5 \times 10^{13}$ cm⁻³.
Growth rate calculations for the ATC parameters

FIG. 3. Frequency (ω) and growth rate (γ0) as a function of \( k \lambda_{De} \) in H\(^+\) plasma. (a) \( \omega_0/\omega_{LH} = 1.57, T_e/T_i = 3, U/C_s = 1, k_0 || \lambda_{De} = 2 \times 10^{-3}, k_\parallel \lambda_{De} = 6 \times 10^{-3} \), \( \omega_{pe}^2/\omega_{ce}^2 = 0.25 \); (b) \( \omega_0/\omega_{LH} = 1.91, T_e/T_i = 1, U/C_s = 1, k_0 || \lambda_{De} = 6 \times 10^{-3} \), \( k_\parallel \lambda_{De} = 1 \times 10^{-2} \), \( \omega_{pe}^2/\omega_{ce}^2 = 0.152 \). For both (a) and (b), \( \omega_{ci}/\omega_0 = 0.033 \).
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FIG. 14. (a) Growth rates for \((\omega_0/\omega_{th}) = 5.05, (T_e/T_i) = 1, (U/C_\infty) = 1\), \((m_i/m_e) = 1840, k_{ii}\lambda_D = 0.022, k_{ii0}\lambda_D = 0.002, (\omega_{pe}/\Omega_e)^2 = 0.020\). (b) \((\omega_0/\omega_{th}) = 2.95, (T_e/T_i) = 1, (U/C_\infty) = 1, (m_i/m_e) = 1840, k_{ii}\lambda_D = 0.012, k_{ii0}\lambda_D = 0.002, (\omega_{pe}/\Omega_e)^2 = 0.0605\).

FIG. 15. (a) Growth rates for \((\omega_0/\omega_{th}) = 1.91, (T_e/T_i) = 1, (U/C_\infty) = 1, (m_i/m_e) = 1840, k_{ii}\lambda_D = 0.006, k_{ii0}\lambda_D = 0.002, (\omega_{pe}/\Omega_e)^2 = 0.152\). (b) Same as in (a), except \(k_{ii}\lambda_D = 0.01, k_{ii0}\lambda_D = 0.006\).
LH PDI thresholds dominated by convective losses out of the resonance cones but can be mitigated by thermal corrections (M. Porkolab, Nuclear Fusion 1978)

For example, for decay into hot ion plasma LH sideband and quasi-modes, the threshold is:

\[
P \text{ (watts)} = \frac{3.76 \times 10^4 \, p_{\text{oy}} c_s^3 B^2 n}{R N_{\text{oll}} c L_{\text{oz}} \omega_o^2 \omega_0} \left( \frac{\pi}{a} \right) \frac{p_1}{p_2} \left( \frac{\omega_o^2}{2} - 1 \right) \left( \frac{\omega_L^2}{\omega_0^2} - 1 - \frac{3}{2} \frac{k^2 v_1^2}{\omega_0^2} \right)^{3/2} \]

where \( p = (\ell_{0z}/\lambda_{0||}) \).

\[\omega^2 = \omega_{LH}^2 \left[ 1 + \frac{k^2}{k^2 m_i} + \frac{3 k^2 \, T_i}{\omega^2 m_e} \left( 1 + \frac{1}{4} \frac{T_e}{T_i} \frac{\omega^4}{\omega_{ce}^2 \omega_{ci}^2} \right) \right] \]

(Bellan and Porkolab, PRL 1975)

Geometry for convective PDI threshold
For decay into ion sound quasi modes
Takase and Porkolab, PF 1983

\[\gamma \Delta x = \frac{\gamma L_z}{|v_{2x}|} \left( v_{2x} - v_{o2x} / v_{o2x} \right) = \pi, \]

Porkolab_RF_2015-Lake Arrowhead-Ca
Takase et al, measured the LH pump wave and PDI sideband with CO$_2$ laser scattering in Alcator C
(Takase, Porkolab, Schuss et al, Phys. Fluids 28, 3, 1985)

FIG. 11. The low-frequency (left) and high-frequency (right) spectra at two different densities. Hydrogen, $B = 8$ T, (a) $\bar{n}_e = 2.1 \times 10^{14}$ cm$^{-3}$ [$\omega_0/\omega_{\text{LH}}(0)$ = 1.6], and (b) $\bar{n}_e = 1.1 \times 10^{14}$ cm$^{-3}$ [$\omega_0/\omega_{\text{LH}}(0)$ = 2.0].

FIG. 12. The frequency-integrated pump power (including the pump broadening) and sideband power (integrated over all sidebands) as functions of density. Deuterium, $B = 8$ T, and $P_{\text{rf}} = 280$ kW.

Does this lead to a density limit for a particular frequency, namely at $\omega_0 < 2\omega_{\text{LH}}$ when PDI becomes dominant? Depends on pump depletion due to PDI?
PDI in Alcator C-Mod LHCD experiments are ubiquitous and complex and are under intense study (S. G. Baek et al, RF Conf. Sorrento, 2013; IAEA, Nucl. Fusion 55, 043009, 2015)

PDI is intensifying as the density is raised so that $\omega < 2 \omega_{\text{LH}}$ near the edge

X ray intensity decreases below theoretical predictions - “density limit” – is it related to the PDI intensifying?
**PDI in C-Mod LHCD Experiments**

(S. G. Baek et al, Nucl. Fusion 55, 043009, 2015)

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**Figure 9.** Pump peak power and the integrated sideband power of the first-harmonic sideband LH waves measured at (a) the launcher, (b) the inner wall and (c) outside of the machine. LSN equilibrium with the field reversal (i.e. $\vec{B} \times \nabla B$ drift direction away the X-point);

$I_p = 550$ kA
Summary of latest paper on PDI in the LHCD regime on C-Mod (S. G. Baek et al, Nucl. Fusion 55, 043009, 2015)

By raising the plasma current, the edge density is modified and X-Ray emission (surrogate for fast electrons and current drive) increases substantially- a potential solution toward raising the PDI threshold and eliminating the density limit – new experiments ongoing at C-Mod

(S. G. Baek et al, NF 55, 043009, 2015)

7. Summary and discussion

In this paper, new results are presented on the role of parametric instabilities in limiting the effectiveness of lower hybrid current drive at high density. While direct proof is still lacking, the circumstantial evidence from these and results from other machines suggests that PDI are involved in the universally observed roll off in LHCD efficiency at high density. The PDI onset observed in Alcator C-Mod are linked to the conditions in the SOL.

Strong PDI observed during high power IBW experiments in DIII-D
(R.I. Pinsker et al, Nuclear Fusion 33, 777(1993))

The non-linear effects have been shown to result in strong edge absorption which was consistent with the lack of observable core heating in these experiments [11, 14]. These results are not inconsistent with the earlier observations on the Alcator C tokamak, which implied strong non-linear effects in a regime of significantly lower ponderomotive forces than in DIII-D.

FIG. 6. (a) The RF spectrum obtained in a deuterium inside-wall limited discharge with $H/(H+D) \approx 2\%$. ($I_p = 0.84$ MA, $\bar{n}_e = 1.9 \times 10^{13}$ cm$^{-3}$, $P_{RF} = 450$ kW, $B_T = 1.7$ T.) (b) The RF spectrum obtained during hydrogen injection into the same discharge as shown in (a).
Renewed interest in “whistler” wave CD (Helicon CD per V. Vdovin)
- see invited talk I-17 by R. Prater at this meeting

 Accessibility to the plasma core depends on N_{\parallel}

LH-CD \quad \omega_0 > \omega(0)_{LH} \quad f_0 = 5 \text{ GHz}

HELICON-CD \quad \omega_0 < \omega(0)_{LH} \quad f_0 = 0.5 – 1.0 \text{ GHz}

Both LHCD \((E_\parallel, E_x)\) and Helicon CD \((E_y)\) are susceptible to PDI near \(\omega_0 = \omega_{LH}\)
- Initial calculations indicate substantial growth rates in DIII-D near \(\omega_0 = \omega_{LH}\)
Summary: Enormous progress in understanding RF physics over the past 50 years

- In the 1970s both linear and nonlinear wave physics information was obtained, and fundamental laboratory experiments verified theoretical predictions.

- To deal with high power RF heating experiments in tokamaks, the theory of parametric instabilities with $k_0 \approx 0$ was developed in the 1960s (Silin) and 1970s (Porkolab and others) to values of $v_{osc}/c_s \geq 1$ in a magnetized plasma.

- PDI has been observed in tokamak experiments in all frequency regimes.

- The practical implications for tokamaks ("density limit", pump depletion, etc) remain to be understood due to difficulties associated with the complex geometry.

- Measuring the PDI amplitudes and radial location with non-perturbative measurements – i.e., CO$_2$ scattering, would be highly desirable.

- PDI in high temperature fusion plasmas will not be important for ECH or ICH.

- Directly launched IBW or EBW probably subject to strong PDI at the edge.

- PDI in LHCD regimes remains to be understood in regard to the "density limit".

- The importance of PDI in the whistler (helicon) regime remains to be tested.